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Strong Convergence of Weighted Sums of Random Elements through the Equivalence of Sequences of Distributions

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The equivalence of sequences of probability measures jointly with the extension of Skorohod's representation theorem due to Blackwell and Dubins is used to obtain strong convergence of weighted sums of random elements in a separable Banach space. Our results include most of the known work on this topic without geometric restrictions on the space. The simple technique developed gives a unified method to extend results on this topic for real random variables to Banach-valued random elements. This technique is also applied to the proof of strong convergence of some statistical functionals. © 1988 Academic Press, Inc.

1. INTRODUCTION

In [6] we have developed a technique, based on the Skorohod almost sure representation theorem [17], to prove Strong Laws of Large Numbers (SLLN) in Banach spaces. However, the use of this technique is conditioned by the fact that, in the Skorohod theorem, a limit probability law for the sequence of random elements in consideration is needed as reference.

In this work a more advantageous version of this rechnique is given by using an extension due to Blackwell and Dubins of the Skorohod representation theorem in conjunction with the concept of weakly equivalent sequences of probabilities.

The paper is mainly devoted to studying the strong stability of weighted sums of random elements in separable Banach spaces without any reference to cancellation conditions (of geometric type). However, in Section 5,

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strong stability of certain statistic functionals will be proved using the same technique.

The basic result in the work is provided by Theorem 1, which establishes the general framework in which all the results considered are proved. This gives a unified method for the extension of known results on this topic concerning real random variables to Banach valued random elements.

Section 3 deals with SLLNs in separable Banach spaces in the general framework of pairwise independent random elements and makes use of the real valued random variable results on this topic in [8, 5]. Our Theorem 3 provides a source for results of the SLLN kind, as shown in Corollary 4 (along the same line as Theorem 2.3 in [7]), in which the Chung condition is imposed, or as Theorem 6, which yields a SLLN considerably more general than that in [19], for sequences of random elements under the conditions of "Césaro-tightness" and stochastic dominance by an integrable random variable.

In Section 4 we consider almost sure convergence of weighted sums, $\sum_{k\geqslant 1} a_{nk} X_k$, of random elements, where the double array $\{a_{nk}, n, k \geqslant 1\}$ of real numbers is a Toeplitz sequence (see [18, p. 63]). The problem in separable Banach spaces has received considerable attention, see, e.g., [13, 20, 22, 18, 3] or [21]. We will prove that once more the scheme provided by Theorem 1 gives simple proofs for more general results than those contained in the literature mentioned.

Finally Section 5 is devoted to explore other possible fields of application of the proposed techniques.

We often follow the notation in [2]: (U, λ) will be a standard probability space, and for each probability, θ, on the Polish space under consideration, ρ_θ will be the Skorohod representation given there. We denote by $L(X)$ to the probability distribution induced by the random element X, EX being its expected value. Also $\operatorname{Var} Z$ will denote the variance of the (real) random variable Z and I_A will be the indicator function of the set A.

2. Basic Result

The key issue in the present work lies in the extension of Skorohod's almost sure representation theorem given by Blackwell and Dubins (see [2]) and applied here to equivalent sequences of probability measures or laws (see [11, p. 373]) on a Polish space.

We say that two sequences $\{P_n\}_n$ and $\{Q_n\}_n$ of laws are weakly equivalent (equivalent in the sequel), and we write $\{P_n\}_n \leftrightarrow \{Q_n\}_n$, if the two sequences have the same weak limit measures for same sequences of subscripts.

Now let $\{P_n\}_n$ and $\{Q_n\}_n$ be sequences of probability measures on a Polish space M. From Prohorov's theorem (see [1, p. 37]) we deduce that if $\{P_n\}_n$ is tight and the sequences $\{P_n\}_n$ and $\{Q_n\}_n$ are equivalent, then $\{Q_n\}_n$ is also tight, and therefore the weak limits obtained along any subsequence are in fact probability measures on the metric space M.

We are in a position to establish the following basic result.

THEOREM 1. *Let $\{P_n\}_n$ and $\{Q_n\}_n$ be equivalent sequences of probability measures on the Polish space M, and let $\{P_n\}_n$ be tight. Then there exist sequences $\{X_n\}_n$, $\{Y_n\}_n$ of M-valued random elements on a standard probability space (U, λ) such that:*

(i) $L(X_n) = P_n$ *and* $L(Y_n) = Q_n$ *for every* n.

(ii) $X_n - Y_n \to 0$ *in* λ*-probability.*

Moreover, if $M = B$ is a separable Banach space and we assume that:

(a) *The mapping* $g(t) = \|t\|$ *is uniformly integrable w.r.t.* $\{Q_n\}_n$, *and*

(b) $\int \|t\| \, dP_n - \int \|t\| \, dQ_n \to 0$; *then:*

(iii) $X_n - Y_n \to 0$ *also in* $L_1(U, \lambda, B)$.

Proof. First recall that the convergence in probability is characterized in terms of the almost sure convergence in the following way:

$f_n \to f$ in λ-probability iff for every subsequence $\{f_{n'}\}_{n'}$ there exists a new subsequence $\{f_{n''}\}_{n''}$ such that $f_{n''} \to f$ λ-a.s.

Now take for X_n and Y_n the representations ρ_{Pn} and ρ_{Qn} obtained in [2], and consider any subsequence of $\{X_n - Y_n\}_n$, say $\{X_{n'} - Y_{n'}\}_{n'}$. Since $\{L(X_{n'})\}_{n'}$ is tight we can obtain a new subsequence $\{L(X_{n''})\}_{n''}$ which converges weakly to a probability measure θ (which obviously depends on the considered subsequence) on M. But then $\{L(Y_{n''})\}_{n''}$ also converges weakly to θ, and the Blackwell–Dubins representations verify:

$$X_{n''} = \rho_{Pn''} \to \rho_\theta \ \lambda\text{-a.s.} \qquad \text{and} \qquad Y_{n''} = \rho_{Qn''} \to \rho_\theta \ \lambda\text{-a.s.}$$

hence $X_{n''} - Y_{n''} \to 0$ λ-a.s., which proves (i) and (ii).

To prove the second part it suffices to show that for any subsequence $\{X_{n'} - Y_{n'}\}_{n'}$ we can obtain a new subsequence $\{X_{n''} - Y_{n''}\}_{n''}$ verifying $X_{n''} - Y_{n''} \to 0$ in $L_1(U, \lambda, B)$.

As above we can choose a subsequence $\{X_{n''}\}_{n''}$ such that $X_{n''} \to \rho_\theta$ λ-a.s. and $Y_{n''} \to \rho_\theta$ λ-a.s.

But by (a) we also have $\int \|Y_{n''}\| \, d\lambda \to \int \|\rho_\theta\| \, d\lambda$, and from (b) $\int \|X_{n''}\| \, d\lambda \to \int \|\rho_\theta\| \, d\lambda$, hence $X_{b''} \to \rho_\theta$ and $Y_{n''} \to \rho_\theta$ in $L_1(U, \lambda, M)$ (see, e.g. [12, p. 233]) so $X_{n''} - Y_{n''} \to 0$ in $L_1(U, \lambda, M)$.

3. SLLN IN BANACH SPACES

The theorem just proved will be used in this section in connection with the sample distribution of a sequence of pairwise independent random elements, for which we establish the following generalization of the Glivenko-Cantelli theorem:

THEOREM 2. *Let $\{X_n\}_n$ be a sequence of pairwise independent random elements defined on the probability space (Ω, σ, P) taking values on a separable metric space, M, and let $\mu_n(\omega, \cdot)$ be the sample distribution based on $X_1(\omega), X_2(\omega), ..., X_n(\omega)$. Then with probability one, the sequence of empirical laws is equivalent to that of Césaro-means of the theoretical laws*:

$$P\left\{\omega/\{\mu_n(\omega, \cdot)\}_n \leftrightarrow \left\{(1/n) \sum_{i \leqslant n} L(X_i)\right\}_n\right\} = 1.$$

Proof. It suffices to show (see [14, p. 47]) that for any bounded and continuous function f on M we have (with $P_i = L(X_i)$):

$$\int f(t)\, \mu_n(\cdot, dt) - (1/n) \sum_{i \leqslant n} \int f(t)\, P_i(dt) \to 0 \qquad P\text{-a.s.}$$

or in an equivalent way

$$(1/n) \sum_{i=1}^{n} [f(X_i) - E(f(X_i))] \to 0 \qquad P\text{-a.s.}$$

This is obvious from the Csörgo–Tandori–Totik extension of Kolmogorov's SLLN (see Theorem 1 in [5]).

This way of considering the Glivenko–Cantelli theorem does not seem to be usual in the literature. However, we would like to emphasize that, for real random variables, it is possible to modify slightly the proof of the Glivenko–Cantelli theorem in [15, pp. 13–16]) to prove this more general version (for mutually independent random variables):

Let $\{X_n\}_n$ be a sequence of independent real random variables defined on the probability space (Ω, σ, P) with distribution functions F_n, and let $G_n(\omega, t)$ be the sample distribution function based on $X_1(\omega), ..., X_n(\omega)$. Then we have

$$P\left\{\omega/\sup_t \left| G_n(\omega, t) - (1/n) \sum_{i \leqslant n} F_i(t) \right| \to 0\right\} = 1.$$

The following theorem yields the basic steps for obtaining a number of

SLLNs in Banach spaces for sequences of pairwise independent random elements.

THEOREM 3. *Let* $\{X_n\}_n$ *be a sequence of pairwise independent random elements taking values on the separable Banach space* B, $E\|X_n\| < \infty$, $n \in N$, *and let* $P_n = L(X_n)$, $n \in N$.

Assume that:

(i) *The sequence of Césaro-means* $\{(1/n)\sum_{i\leqslant n} P_i\}_n$ *is tight.*

(ii) *The sequence* $\{\|X_n\|\}_n$ *verifies the SLLN* (*i.e.*,

$$(1/n)\sum_{i\leqslant n}(\|X_i\| - E\|X_i\|) \to 0, \ P\text{-}a.s.)).$$

(iii) *The mapping* $g(t) = \|t\|$ *is uniformly integrable w.r.t.* $\{(1/n)\sum_{i\leqslant n} P_i\}_n$.

Then $\{X_n\}_n$ *verifies the SLLN*:

$$(1/n)\sum_{i\leqslant n}(X_i - EX_i) \to 0 \qquad P\text{-}a.s.$$

Proof. Denote, as above, $\mu_n(\omega, \cdot)$ the sample distribution and define $\Omega^* = \{\omega \in \Omega, (1/n)\sum_{i\leqslant n}[\|X_i(\omega)\| - E\|X_i\|] \to 0$ and $\{(1/n)\sum_{i\leqslant n} P_i\}_n \leftrightarrow \{\mu_n(\omega, \cdot)\}_n\}$.

By (ii) in the assumptions and Theorem 2: $P(\Omega^*) = 1$.

From now on suppose that $\omega \in \Omega^*$ and denote respectively by U_n^ω and V_n the representations, defined on the probability space (U, λ), obtained in Theorem 1 for the probability measures $\mu_n(\omega, \cdot)$ and $(1/n)\sum_{i\leqslant n} P_i$ respectively.

Also note that our assumptions (ii) and (iii) coincide with (a) and (b) in Theorem 1, and hence $U_n^\omega - V_n \to 0$ in $L_1(U, \lambda, B)$ and in particular $\int U_n^\omega \, d\lambda - \int V_n \, d\lambda \to 0$ for every $\omega \in \Omega^*$.

This proves the SLLN for the sequence $\{X_n\}_n$.

For simplicity we often use the term "tight" for a sequence of random elements $\{X_n\}_n$ when the sequence $\{L(X_n)\}_n$ is tight.

In [7] (also in [9]) the condition called "compact uniform integrability" is used in lieu of tightness for a sequence $\{X_n\}_n$ of independent random elements to provide a SLLN under the additional hypothesis of Chung's condition. But after Proposition 2.5 in [7] it is obvious to see that in separable Banach spaces the condition of compact uniform integrability for the sequence $\{X_n\}_n$ is equivalent to tightness together with uniform integrability. Therefore our assumptions (i) and (iii) are weaker than the compact uniform integrability for the sequence $\{X_n\}_n$. The following example (kindly provided by a referee) shows this fact even for real random variables:

Let $\{X_n\}_n$ be defined by $X_n = 0$ if $n \neq \frac{1}{2}(k(k+1))$ for some integer k, and $X_n = \log k$ if $n = \frac{1}{2}(k(k+1))$ for some integer k. Then the sequence $\{X_n\}_n$ verifies (i) and (iii) in the hypothesis of Theorem 3 but it is not compact uniformly integrable (in fact the sequence is not tight).

Also note that for independent random elements, assumption (ii) is verified under the Chung condition: $\sum_{n \geqslant 1} n^{-p} E \|X_n\|^p < \infty$ for some $1 \leqslant p \leqslant 2$. More generally, from Theorem 2 in [5], it is easy to show that Chung's condition implies our assumption (ii) for pairwise independent random elements under the additional hypothesis that $(1/n) \sum_{i \leqslant n} E \,| \|X_i\| - E \|X_i\| \,| = O(1)$. But this hypothesis is an easy consequence of the uniform integrability of $\{\|X_n\|\}_n$. Hence we can establish a more general version of Theorem 2.3 in [7] as a corollary of Theorem 3:

COROLLARY 4. *Let $\{X_n\}_n$ be a sequence of pairwise independent random elements in the separable Banach space B satisfying*:

(i) *$\{X_n\}_n$ is compactly uniformly integrable*;

(ii) *$\sum_{n \geqslant 1} n^{-p} E \|X_n\|^p < \infty$ for some $1 \leqslant p \leqslant 2$.*

Then $\{X_n\}_n$ verifies the SLLN:

$$(1/n) \sum_{i \leqslant n} (X_i - EX_i) \to 0 \qquad a.s.$$

It is well known that assumption (iii) in Theorem 3 will be automatically satisfied in the hypothesis of boundedness: $\sup_n \|X_n\|^{1+\delta} < \infty$ for some $\delta > 0$ (considered in [19]); or also if $P\{\|X_n\| \geqslant a\} \leqslant P\{|Z| \geqslant a\}$, $a > 0$, $n \in N$ for some integrable real-valued random variable Z. In fact, the last condition of stochastic dominance by an integrable random variable is less restrictive than that of boundedness (see, e.g., [18, Lemma 5.2.2]).

Now the hypothesis of stochastic dominance by an integrable random variable also guarantees that assumption (ii) in Theorem 3 holds. This is merely an exercise after Theorem 2 in [5] and the following lemma:

LEMMA 5. *Let $\{X_n\}_n$ be a sequence of positive real random variables stochastically dominated by an integrable random variable Z. Then*

$$\sum_{n=1}^{\infty} \mathrm{Var}(X_n I_{(X_n \leqslant n)})/n^2 < \infty.$$

Proof. Obviously it suffices to show that $\sum_{n \geqslant 1} E[X_n^2 I_{(X_n \leqslant n)}]/n^2 < \infty$. To do this, take into account that

$$\begin{aligned}
E[X_n^2 I_{(X_n \leqslant n)}] &= \int_0^{n^2} P\{X_n > \sqrt{t}\}\, dt - n^2 P\{X_n > n\} \\
&\leqslant \int_0^{n^2} P\{Z > \sqrt{t}\}\, dt - n^2 P\{X_n > n\} \\
&= E[Z^2 I_{(Z \leqslant n)}] + n^2 [P\{Z > n\} - P\{X_n > n\}] \\
&\leqslant E[Z^2 I_{(Z \leqslant n)}] + n^2 P\{Z > n\}.
\end{aligned}$$

Therefore

$$\sum_{n \geqslant 1} E[X_n^2 I_{(X_n \leqslant n)}]/n^2 \leqslant \sum_{n \geqslant 1} E[Z^2 I_{(Z \leqslant n)}]/n^2 + \sum_{n \geqslant 1} P\{Z > n\}$$

and the series in the right-hand side of the inequality are convergent because of $EZ < \infty$.

The previous considerations prove the following new consequence of Theorem 3:

THEOREM 6. *Let $\{X_n\}_n$ be a sequence of pairwise independent random elements in the separable Banach space B, such that:*

(i) *The sequence $\{(1/n) \sum_{i \leqslant n} L(X_i)\}_n$ is tight, and*

(ii) *The sequence $\{\|X_n\|\}_n$ is stochastically bounded by an integrable random variable Z (i.e., $P\{\|X_n\| \geqslant a\} \leqslant P\{Z \geqslant a\}$, $a > 0$, $n \in N$).*

Then $\{X_n\}_n$ verifies the SLLN:

$$(1/n) \sum_{i \leqslant n} (X_i - EX_i) \to 0 \qquad P\text{-a.s.}$$

Theorem 6 and, of course, Theorem 3 feature among the few SLLN-like theorems in normed linear spaces which do not require any cancellation condition on the space. The improvements of Theorem 3 on the available results of this kind we are aware of are patent. On the one hand, only Etemadi's version of the SLLN of Mourier (see [8]) or the generalization of the results of Jamison, Orey and Pruitt given in [21] as Theorems 4 and 5 consider sequences of pairwise independent random elements instead of the common assumption of mutual independence. On the other hand, the stronger closest result in the literature (for sequences of mutually independent random elements, of course) is probably given in the above-mentioned Theorem 2.3 in [7]. The advantages of our Theorem 3 on this result, even in the case of mutual independence hypotheses, are twofold. First, the assumption of tightness in Daffer and Taylor is replaced by the weaker one of "Césaro-tightness" or "tightness in mean" (the same observation works for the assumption of uniform integrability). Moreover, the assumption of Chung's condition is obviously more rigid than (ii) in our Theorem 3 (practically necessary). For example, our Theorem 6 is not a consequence of Chung's condition.

4. Stability of Weighted Sums of Random Elements

In this section we are concerned with the a.s. convergence to 0, as $n \to \infty$, of weighted sums, $\sum_{k \geq 1} a_{nk}(X_k - EX_k)$, of random elements in the separable Banach space B. Here $(a_{nk})_{n,k \geq 1}$ is a Toeplitz sequence (T-sequence) of real numbers, i.e.:

$$\text{(A)} \quad \lim_{n \to \infty} a_{nk} = 0 \qquad \text{for each } k \geq 1$$

and

$$\text{(B)} \quad \sum_{k \geq 1} |a_{nk}| \leq C \qquad \text{for each } n \geq 1.$$

From now on assume w.l.o.g. that $\sum_{k \geq 1} |a_{nk}| \leq 1$ for each n. Also, for $k \geq 1$, let us respectively denote by a_{nk}^+ and a_{nk}^- the positive real numbers $a_{nk}^+ = \sup\{a_{nk}, 0\}$, $a_{nk}^- = \sup\{-a_{nk}, 0\}$.

A common requirement on the involved sequence $\{X_n\}_n$ will be that $E\|X_k\| \leq M$, $k \geq 1$; hence the almost sure absolute convergence of the series $\sum_{k \geq 1} a_{nk}(X_k - EX_k)$ follows from the fact that $\sum_{k \geq 1} |a_{nk}| \, \|X_k - EX_k\| < \infty$.

In fact, the associative property of absolute convergent series in Banach spaces permits us to consider separately positive and negative coefficients a_{nk}, and to handle independently the two series: $\sum_{k \geq 1} a_{nk}^+ X_k$ and $\sum_{k \geq 1} a_{nk}^- X_k$, taking into account that $\sum_{k \geq 1} a_{nk} X_k = \sum_{k \geq 1} a_{nk}^+ X_k + \sum_{k \geq 1} a_{nk}^- X_k$.

More precisely, including the random element $X_0 \equiv 0$ and convenient weights $a_{n0}^+ = 1 - \sum_{k \geq 1} a_{nk}^+$ and $a_{n0}^- = 1 - \sum_{k \geq 1} a_{nk}^-$, it is obvious that our problem becomes that of studying the a.s. convergence: $T_n = \sum_{k \geq 0} a_{nk}(X_k - EX_k) \to 0$, as $n \to \infty$, where $(a_{nk})_{nk}$ verifies:

$$\text{(i)} \quad a_{nk} \geq 0 \qquad \text{for each } n \geq 1, k \geq 0$$

$$\text{(ii)} \quad \lim_{n \to \infty} a_{nk} = 0 \qquad \text{for each } k \geq 1$$

and

$$\text{(iii)} \quad \sum_{k \geq 0} a_{nk} = 1 \qquad \text{for each } n \geq 1.$$

This viewpoint and once more Theorem 1 are the basic tools we use to extend the known real random variables results on this topic to Banach valued random elements. For let $\mu_n(\omega, \cdot)$ be the probability measure giving mass a_{nk} to $X_k(\omega)$, and let λ_n be the probability defined by $\lambda_n = \sum_{k \geq 0} a_{nk} P_k$, where $P_k = L(X_k)$, $k \geq 0$. Then $T_n(\omega) = \int t\mu_n(\omega, dt) - \int t\lambda_n(dt)$, and Theorem 1 will provide the desired result on the basis that:

(1) $\{\lambda_n\}_n$ is tight.

(2) The mapping $g(t) = \|t\|$ is uniformly integrable w.r.t. $\{\lambda_n\}_n$.

(3) $\int \|t\| \, \mu_n(\omega, dt) - \int \|t\| \, \lambda_n(dt) \to 0$ for P-a.e. ω.

(4) $P\{\omega / \{\mu_n(\omega, \cdot)\}_n \leftrightarrow \{\lambda_n\}_n\} = 1$.

Now let us consider separately each point in terms of the original T-sequence verifying (A) and (B).

Part (1) will be automatically satisfied under each one of the assumptions:

1.1 $\{X_n\}_n$ is a sequence of identically distributed random elements.

1.2. $\{X_n\}_n$ is tight.

1.3. The series $\sum_{k \geq 1} |a_{nk}|$ converge uniformly in n (i.e., $\lim_{m \to \infty} \sum_{k \geq m} |a_{nk}| = 0$ uniformly in n).

Note that condition 1.3 is equivalent to the assumption that the sequences, $\{Q_n\}_n$ and $\{R_n)_n$, of probability measures on the positive integers defined by $Q_n(k) = a_{nk}^+$, $R_n(k) = a_{nk}^-$ $k = 0, 1, ..., n, ..., n \geq 1$, are tight. This obviously implies (1).

Condition 1.3 holds in particular when:

1.4. $\lim_{n \to \infty} \sum_{k \geq 1} |a_{nk}| = 0$

Part (2) is true in the following cases:

2.1. The random elements X_n, $n \geq 1$, are identically distributed and $E \|X_1\| < \infty$.

2.2 $\{X_n\}_n$ is uniformly integrable, in particular in the following case.

2.3. The sequence $\{\|X_n\|\}_n$ is stochastically bounded by an integrable random variable Z.

Obviously (3) holds iff

$$(3') \quad \sum_{k \geq 1} a_{nk}(\|X_k\| - E\|X_k\|) \to 0 \quad \text{a.s.}$$

and condition (4) is equivalent to

(4') $\sum_{k \geq 1} a_{nk}(f(X_k) - Ef(X_k)) \to 0$ a.s. for every continuous and bounded real function f.

In other words, we are in a position to prove the a.s. stability of the sums $\sum_{k\geqslant 1} a_{nk}(X_k - EX_k)$, where $(a_{nk})_{n,k\geqslant 1}$ is a T-sequence verifying (A) and (B) and $\{X_k\}_k$ is a sequence of random elements in B, in virtually all the cases (without geometric restrictions) considered in the available literature.

We mention here the most relevant consequences of these considerations.

For identically distributed random elements $\{X_n\}_n$, assuming $E\|X_1\| < \infty$, conditions 1.1 and 2.1 hold, hence $\sum_{k\geqslant 0} a_{nk}(X_k - EX_k) \to 0$ a.s. in each case in which some result for real random variables implies (3′) and (4′). This happens in particular in Theorems 4 and 5 in [21], extending Etemadi's SLLN to the framework of the Jamison, Orey and Pruitt results [10]. This is also the case in the extensions of Pruitt's theorem by Padgett and Taylor (see, e.g., [18, Theorem 5.1.3]) and by Bozorgnia and Bhaskara Rao [3, Theorem 3.2].

For the nonidentically distributed case, the paradigm is given by Rohatgi's result for independent real random variables [16]:

Let $\{X_n\}_n$ be a sequence of independent real random variables stochastically dominated by the real random variable X. Let $\{a_{nk}\}_{n,k\geqslant 1}$ be a T-sequence verifying $\max_{k\geqslant 1}|a_{nk}| = O(n^{-\alpha})$ for some $\alpha > 0$. If $E|X|^{1+1/\alpha} < \infty$ then

$$\sum_{k\geqslant 0} a_{nk}(X_k - EX_k) \to 0 \qquad a.s.$$

Our extension of Rohatgi's result to Banach spaces is given by:

THEOREM 7. *Let $\{a_{nk}\}_{n,k\geqslant 1}$ be a T-sequence verifying $\max_{k\geqslant 1}|a_{nk}| = O(n^{-\alpha})$ for some $\alpha > 0$, and let $\{X_n\}_n$ be a sequence of random elements in the separable Banach space B such that:*

(i) *The sequence $\{\|X_n\|\}_n$ is stochastically bounded by a real random variable Z with $E|Z|^{1+1/\alpha} < \infty$ and*

(ii) *Any of conditions 1.1 to 1.4 is satisfied.*

Then $\sum_{k\geqslant 0} a_{nk}(X_k - EX_k) \to 0$ a.s.

Theorem 7 contains all the known extensions of Rohatgi's result to Banach spaces (without geometric restrictions on the space). In particular we mention Theorems 5.2.7 and 5.2.8 in [18] and Theorem 2.2 in [3].

5. Application to Statistical Functionals

A careful reading of the proofs of the results developed in the preceeding sections shows that there exist consequences of the employed techniques that remain still unexploited.

To be more precise, let $r \geqslant 1$ and replace assumptions (ii) and (iii) in Theorem 3 by

(ii*) The sequence $\{\|X_n\|^r\}_n$ verifies the SLLN and

(iii*) The mapping $g(t) = \|t\|^r$ is uniformly integrable w.r.t. $\{(1/n) \sum_{i \leqslant n} P_i\}_n$.

The proof of Theorem 3 above works to prove that (with the notation employed there);

(a) Every subsequence $\{V_{n'}\}_n$ of $\{V_n\}_n$ possesses a new subsequence $\{V_{n''}\}_{n''}$ such that for some random variable ρ_θ: $V_{n''} \to \rho_\theta$ in $L_r(U, \lambda, B)$ and

(b) For every ω in a probability-one set and same subscripts as in (a): $U^\omega_{n''} \to \rho_\theta$ in $L_r(U, \lambda, B)$.

Consequently for an L_r-continuous statistical functional T valued on a metric space (M, d) we have $d[T(\mu_{n''}(\omega, \cdot), T((1/n'') \sum_{i \leqslant n''} P_i)] \to 0$, hence $d[T(\mu_n(\omega, \cdot), T((1/n) \sum_{i \leqslant n} P_i)] \to 0$ for p-a.e. ω.

A typical example of L_r-continuous statistical functional is the r-mean, $r > 1$, of a probability measure on a uniformly convex Banach space (see, e.g., [6]). As an example, we give below the analogues of Theorem 6 for these centralization measures:

THEOREM 8. *Let $\{X_n\}_n$ be a sequence of pairwise independent random elements in the separable uniformly convex Banach space B such that*:

(i) *The sequence $\{(1/n) \sum_{i \leqslant n} P_i\}_n$ is tight, and*

(ii) *The sequence $\{\|X_n\|^r\}_n$, $r > 1$, is stochastically bounded by an integrable random variable Z.*

Denote by $m_n(\omega)$ the sample r-mean bsed on $X_1(\omega), ..., X_n(\omega)$, and let Π_n the r-mean of the law $(1/n) \sum_{i \leqslant n} L(X_i)$. Then

$$\|m_n(\omega) - \Pi_n\| \to 0 \qquad \textit{for} \quad P\text{-a.e. } \omega.$$

Theorem 8 generalizes Theorem 4 in [6].

Let us remark that even for real valued random variables one infrequently finds results relating the asymptotic behavior of the sample statistic functionals to that of the original sequence with the exceptions of the case of the mean and the case of independent, identically distributed variables. In the sense, the technique just developed is probably pioneering in this field.

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